Table 2 Calorimetric total hemispherical emittances, with probable errors, of vacuum-deposited gold films

470 Å		Gold on Mylar 550 Å[a]		720 Å[b]		Gold on Kapton 400 Å[c]	
$\epsilon_s \times 10^4$	°K	$\epsilon_s \times 10^4$	°K	$\epsilon_s \times 10^4$	°K	$\epsilon_s \times 10^4$	°K
109 ± 8	61	114 ± 10	63	83 ± 7	63	188 ± 9	196
119 ± 3	88	129 ± 4	91	102 ± 3	90	204 ± 10	255
131 ± 4	129	152 ± 5	129	120 ± 2	132	212 ± 11	314
154 ± 5	162	156 ± 5	149	132 ± 4	166		
203 ± 6	253	212 ± 6	253	175 ± 5	259		
220 ± 7	307	244 ± 8	302	210 ± 6	312		
Gold on Copper							
400 Å[d]		520 Å[d]		4000 Å[d]		2500 Å[c]	
$\epsilon_s \times 10^4$	°K	$\epsilon_s \times 10^4$	°K	$\epsilon_s \times 10^4$	°K	$\epsilon_s \times 10^4$	°K
97 ± 8	59	108 ± 9	59	85 ± 7	59	135 ± 7	201
121 ± 4	96	118 ± 4	96	96 ± 3	97	153 ± 8	255
158 ± 5	161	153 ± 5	163	118 ± 4	161	164 ± 8	302

[a] Thickness measured on film samples, adjacent to emittance specimen, 200 Å thickness variation observed across sheet, ±100 Å.
[b] Thickness measured on 1-in. square area from center of emittance specimen, ±50 Å.
[c] Thickness per supplier, measurement method not known.
[d] Thickness measured on monitors adjacent to emittance specimen, ±50 Å.

quality and type of the substrate material, the purity of the film, and the vacuum and the rate under which the film was deposited. The effects of the first three items could be neglected when discussing ϵ for the 2500 Å thick sample. The smoothness and the flatness of the film surface and the substrate will affect the spectral measurements much more than the total measurements. Also, the ratio of the length characterizing the roughness of the surface to the characteristic wavelength of the incident radiation will decrease rapidly as the temperature decreases. Thus, the effect of roughness is usually negligible at cryogenic temperatures. The effect of sample purity, for purities higher than 99.99%, was also found to be negligible.[13] The remaining item, i.e., the vacuum and the rate at which the film was deposited, however, is very important. For example, the spectral emissivity of metallic films deposited at a rate of 7 Å/sec and under high-vacuum conditions, 10^{-5} torr, could be two or three times higher than the emissivity of a similar film deposited under ultra-high-vacuum conditions, 10^{-9} torr. The rate at which the film is deposited is also an important factor. The higher the rate of deposition, the better and purer is the film. Measurements determining the effect of deposition rate on the film emissivity, however, are not available.

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Changes in the Vibration Characteristics due to Changes in the Structure

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Nomenclature

a	=	see Eq. (10)
$F, \Delta F$	=	flexibility matrix and change in F, respectively
I	=	unit matrix
$M, \Delta M$	=	mass matrix and change in M, respectively
$Q(\lambda_r)$	=	"adjoint" matrix, see Eq. (9)
$U, \Delta U$	=	dynamic matrix and change in U, respectively
$x, \Delta x$	=	eigenvector and change in x, respectively
y	=	eigenrow
$\lambda, \Delta\lambda$	=	eigenvalue and change in λ, respectively
λ_A	=	$\lambda + \Delta\lambda$
ω	=	circular frequency (rad/sec)
$\lfloor\ \rfloor$	=	row matrix
$\{\ \}$	=	column matrix
$[\ ]$	=	square matrix

Introduction

THE influence of small changes in a structure on its frequencies and mode shapes has been discussed by Krishna Murty and Viswa Murty.[1] They consider the following problem: the equations of motion of undamped vibrations of a system can be written as

$$([I] - \omega^2[F][M])\{x\} = 0 \tag{1}$$

When $[F]$ and $[M]$ are changed to $[F + \Delta F]$ and $[M + \Delta M]$, they obtain expressions for $\Delta\omega^2$ and $\{\Delta x\}$ (the changes in the frequency and the mode shape) in terms of $[F]$, $[M]$, $[F + \Delta F]$, and $\{x\}$.

In the following sections, a simple procedure (method A) will first be given, which permits one to obtain quick estimates of the changes of the frequency and the mode shape. A more accurate method presented by Morgan[2] (method B) is de-

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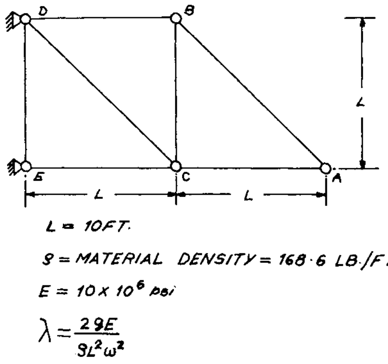

Fig. 1 A plane pin-joined truss.

scribed next. It gives a more accurate estimate of the change in the frequency due to a change in the dynamic matrix $[U]$.

Method A

Equation (1) can be written as

$$[U]\{x\} = \lambda\{x\} \tag{2}$$

where $[U]$ is the (nondimensional) dynamic matrix and λ is the nondimensional frequency parameter.

Let, as a result of change $[\Delta F]$ and $[\Delta M]$ in $[F]$ and $[M]$, the matrix $[U]$ be changed to $[U + \Delta U]$ and λ and $\{x\}$ to (λ_A) and $\{x_A\}$. To obtain λ_A and $\{x_A\}$ it is necessary to solve the equation

$$[U + \Delta U]\{x_A\} = \lambda_A\{x_A\} \tag{3}$$

An estimate for $\{x_A\}$ and λ_A can be obtained if we neglect $\{\Delta x\}$ on the left-hand side and write

$$\lambda_A\{x_A\} = [U + \Delta U]\{x\} \tag{4}$$

The normalizing factor for the product $[U + \Delta U]\{x\}$ gives an approximate estimate for λ_A. (In fact, this is the first step in the iteration for λ_A and $\{x_A\}$. If the iteration process is continued it should converge to the values corresponding to the fundamental mode.)

This procedure was applied to the plane truss analyzed in Ref. 1. For reference, this truss is shown in Fig. 1. Table 1 gives a comparison of the values of λ_A and $\{x_A\}$ obtained from Eq. (4) with those obtained by the exact method [i.e., by the solution of Eq. (4)] and by the method of Ref. 1 (as well as method B, discussed later).

It is surprising to see that the values obtained by using Eq. (4) are in closer agreement with the exact values of both λ_A and $\{x_A\}$, even up to a change of 50% in the area of cross section of the member AB, than are the values obtained by the method of Ref. 1.

Method B

In Eq. (2), when $[U]$ is changed to $[U + \Delta U]$, the corresponding change in a simple eigenvalue λ_r can be estimated from a relationship presented by Faddeev[3]:

$$\Delta\lambda_r = \lfloor y_r \rfloor [\Delta U]\{x_r\} \tag{5}$$

where $\{x_r\}$ is an eigenvector and $\lfloor y_r \rfloor$ is an eigenrow, satisfying the equations

$$[U]\{x_r\} - \lambda_r\{x_r\} = 0, \lfloor y_r \rfloor [U] - \lambda_r \lfloor y_r \rfloor = 0$$
$$\{x_r\}\lfloor y_r \rfloor = 1 \tag{6}$$

Eq. (5) can be rewritten as

$$\Delta\lambda_r = [x_r \cdot y_r] * [\Delta U] \tag{7}$$

where $*$ denotes the inner product of two matrices, i.e.,

$$[A]*[B] = \Sigma a_i \cdot b_i$$

and a_i represents the ith row of A and b_i represents the ith row of B. Morgan[2] shows that, for $\Delta\lambda_r \to 0$, $[x_r \cdot y_r]$ can be expressed as

$$[x_r \cdot y_r] = \{\text{Trace } [Q(\lambda_r)]\}^{-1}[Q(\lambda_r)] \tag{8}$$

where $Q(\lambda_r)$ is given by

$$[Q(\lambda_r)] = [Q_n]\lambda_r^{n-1} + [Q_{n-1}]\lambda_r^{n-2} + \ldots + [Q_1] \tag{9}$$

and is called the "adjoint" matrix.[4] Equation (8) holds only for the case when λ_r is a simple root (i.e., $\lambda_i \neq \lambda_r$).

Rosenbrock[5] gives an expression for $Q(\lambda_r)$ that depends on all the eigenvalues of the matrix $[U]$. Morgan[2] points out that it is more efficient to use Leverrier's algorithm for the determination of $[Q(\lambda_r)]$.[4] For $j = 1, 2, \ldots, n$, this algorithm is

$$a_n = 1 \quad [Q_n] = [I]$$
$$a_{n-j} = -(1/j) \text{ Trace } ([U][Q_{n-j+1}]) \tag{10}$$
$$[Q_{n-j}] = a_{n-j}[I] + [U][Q_{n-j+1}]$$

The matrix $[Q_0]$ must be zero, and so it provides an efficient check on the computation. From Eqs. (7) and (8), we get $\Delta\lambda_r$ as

$$\Delta\lambda_r = \{\text{Trace } [Q(\lambda_r)]\}^{-1}[Q(\lambda_r)]*[\Delta U] \tag{11}$$

In the last column of Table 1 are presented the values of $\lambda_A = (\lambda_r + \Delta\lambda)$ obtained by this method for the plane truss. In all the cases it can be seen that this method (B) gives better estimates for the eigenvalue than the other two methods.

Discussion of the Results and Conclusions

For the problem considered, the values of the eigenvalue λ_A are always greater than the true values when calculated by the method of Ref. (1) and also by either of the methods enumerated here. That is, all the three methods give underesti-

Table 1 Frequencies and mode shapes of the pin-jointed truss of Fig. 1

Area of AB	Direct method λ^b	Direct method $x_A, x_B, x_C{}^a$	Method of Ref. 1 λ^b	Method of Ref. 1 $x_A, x_B, x_C{}^a$	Method A, Eq. (4) λ	Method A, Eq. (4) x_A, z_B, x_C	Method B λ
A	54.041	1, 0.613, 0.530	54.041	1, 0.613, 0.530	54.041	1, 0.613, 0.530	54.041
0.9A	52.400	1, 0.607, 0.526	52.400	1, 0.607, 0.526	52.467	1, 0.608, 0.526	52.425
0.8A	50.499	1, 0.599, 0.520	51.116	1, 0.599, 0.520	50.964	1, 0.601, 0.522	50.969
0.7A	48.710	1, 0.590, 0.513	49.891	1, 0.587, 0.511	49.581	1, 0.593, 0.516	49.333
0.6A	46.522	1, 0.577, 0.503	48.710	1, 0.570, 0.497	48.068	1, 0.586, 0.511	48.020
0.5A	45.612	1, 0.588, 0.488	48.137	1, 0.545, 0.508	46.536	1, 0.578, 0.506	46.269

[a] Taken from Ref. 1, Table 1.
[b] Calculated from Ref. 1.

mates of the frequencies. (In Table 1, λ is given by

$$\lambda = 2gE/\zeta L^2\omega^2 \tag{12}$$

with $g = 32.2$ ft/sec^2, $E = 144 \times 10^7$ lb/ft^2, $\zeta = 168.6$ lb/ft^3, and $L = 10$ ft.)

When the area of the member AB is reduced to half its original value, the frequency (ω) is increased by 8.8%. The values of the increase predicted by the method of Ref. 1 is 5.96%, that predicted by method A is 7.8%, and that predicted by the more accurate method B is 8.1%.

From Table 1 it can be seen that for reductions of the area of AB up to 30% the mode shapes predicted by method A are in good agreement with the values obtained by the exact method. Thus, it appears that method A can be used with confidence to predict the changes in the eigenvalues and the eigenvectors of matrix resulting from small changes in the matrix. It should be noted that this method is strictly valid only for the fundamental frequency.

Method B gives better estimates of the frequencies than either method A or the method of Ref. 1. There are two advantages in this method: a) the matrix $[Q(\lambda_r)]$ need be calculated only once and can be used to predict the changes in λ_r resulting from a change in $[U]$, and b) it can be used to predict the change in any (simple) eigenvalue and is not restricted to the fundamental mode only.

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